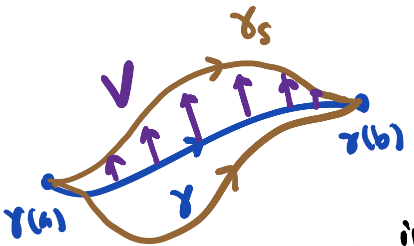
MATH 5061 Lecture 11 (Mar 31)

Recall: $\gamma : [a,b] \to (M^n, g)$ geodesic (i.e. $E'(0) = 0$ $\forall$ variation γ_s) — $\forall$: fixing end pts



2nd variation for E: $V := \frac{\partial \gamma_s}{\partial s}\Big|_{s=0}$ "variation vector field"

$$E''(0) = \int_a^b \left[\|\nabla_{\gamma'} V\|^2 - \langle R(\gamma', V)\gamma', V \rangle \right] dt$$

$\leadsto$ index form $\quad I(V, W) := \int_a^b \langle \nabla_{\gamma'} V, \nabla_{\gamma'} W \rangle - \langle R(\gamma', V)\gamma', V \rangle dt$

$\leadsto$ look at the "kernel" of this symmetric bilinear form

Jacobi fields:

$$\boxed{\nabla_{\gamma'} \nabla_{\gamma'} V + R(\gamma', V)\gamma' = 0}$$

2nd order linear ODE system.

$$V = \underbrace{\overset{\text{tangent to } \gamma'}{V^T}}_{\text{SS}} + \underbrace{\overset{\text{normal to } \gamma'}{V^N}}$$

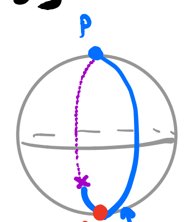
V^T: reparametrization of γ (linear function of γ' in t)

V^N: contain information about the geometry of (M^n, g)

Conjugate points & minimizing geodesics

Recall: "Gauss Lemma": $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle \equiv 1$ in geodesic normal coord. $\Rightarrow$ "short" geodesics are (length / energy) minimizing

E.g.) S^n



Conjugate pt to p

fail to be minimizing after passing the antipodal pt.

Q: What about the "long" geodesics?

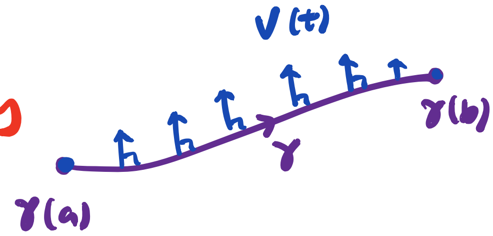
A: related to normal Jacobi fields!

Idea: A geodesic will fail to be minimizing once it passes through a "conjugate pt.".

Def: Let $\gamma : [a,b] \to (M^n, g)$ be a geodesic.

We say that $\gamma(a)$ & $\gamma(b)$ are **conjugate** (along γ) if $\exists$ non-trivial Jacobi field $0 \not\equiv V(t)$ along γ st

$$V(a) = 0 = V(b)$$



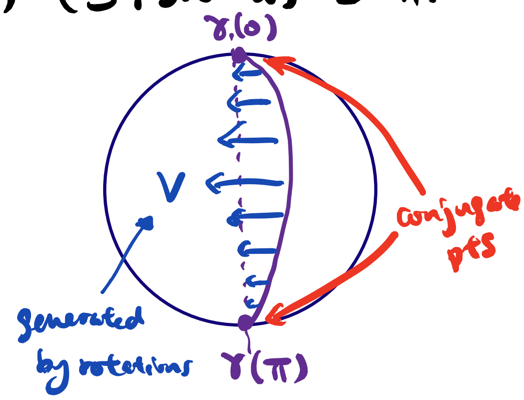
Furthermore, we define the **multiplicity** as the dimension of the vector space of all such V above.

Remark: Since V^T is a linear function in t (times γ')

vanishing of V^T at end pts $\Rightarrow$ $V^T \equiv 0$ (ie the V above must be normal to γ')

So, multiplicity $\leq n-1$

E.g.) $(S^n, g_{round}) \subseteq \mathbb{R}^{n+1}$.



$\gamma : [0,\pi] \to (S^n, g_{round})$ great circle joining antipodal pts.

Any rotation in $\mathbb{R}^{n+1}$ fixing $\gamma(0), \gamma(\pi)$ generates a Jacobi field V vanishing at the end points

$\Rightarrow$ multiplicity $= n-1$

There is a relation between "conjugate points" & the "singularities of the exponential map"

Prop: Let $\gamma : [0,1] \to (M^n, g)$ be a geodesic joining $p = \gamma(0)$, $q = \gamma(1)$.

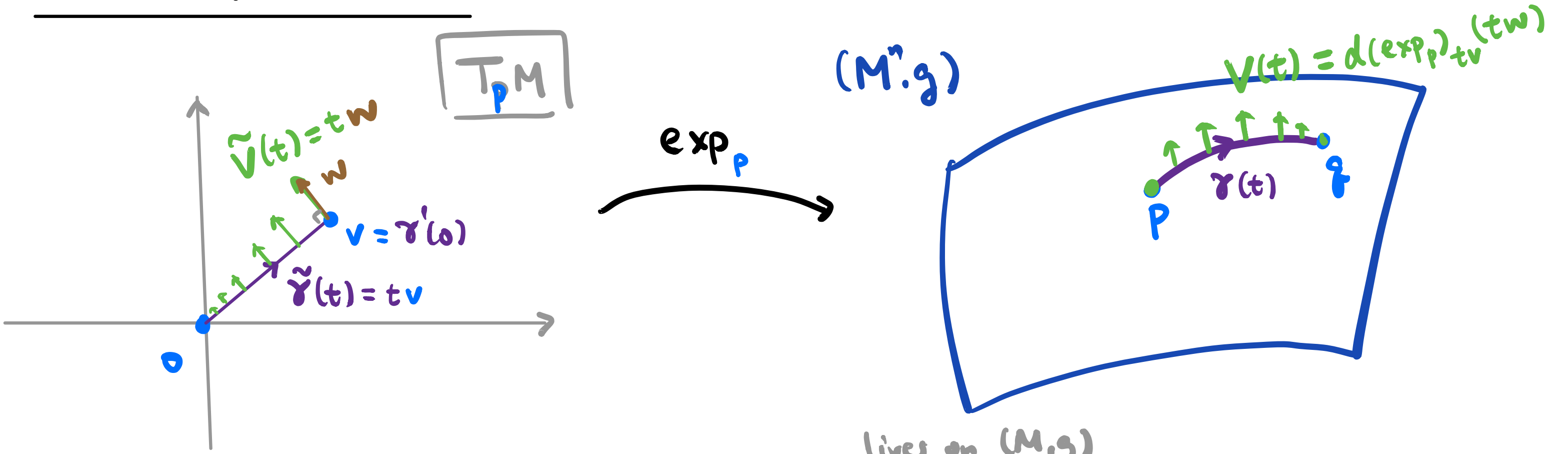
WLOG

THEN, (1) p, q are conjugate along γ
$\Updownarrow$
(2) $v = \gamma'(0) \in T_pM$ is a "critical pt." of $\exp_p$

ie $d(\exp_p)_v$ is singular

Furthermore, multiplicity = $\dim(\ker(d(\exp_p)_v))$.

"IDEA of the Proof":



Observation: Any normal Jacobi field $V(t)$ along γ st $V(0) = 0$

lives on (M, g)

has the form $V(t) = d(\exp_p)_{tv}(tw)$ for some $w \in v^\perp \subseteq T_pM$

Note: $w = V'(0)$

V is a Jacobi field vanishing at the end pts $\iff$ $V(1) = d(\exp_p)_v(w) = 0 \in T_qM$.

ie $w \in \ker(d(\exp_p)_v)$.

$\square$

Thm: Let $\gamma : [0,1] \to (M^n, g)$ be a geodesic with $p = \gamma(0)$, $q = \gamma(1)$.

(i) If $\gamma(t)$ is NOT conjugate to $\gamma(0) = p$ for all $t \in [0,1]$,

then γ is a "locally" minimizing geodesic between the end pts.

ie $\forall$ curve $\alpha \overset{C^0}{\approx} \gamma$, $L(\alpha) \geq L(\gamma)$

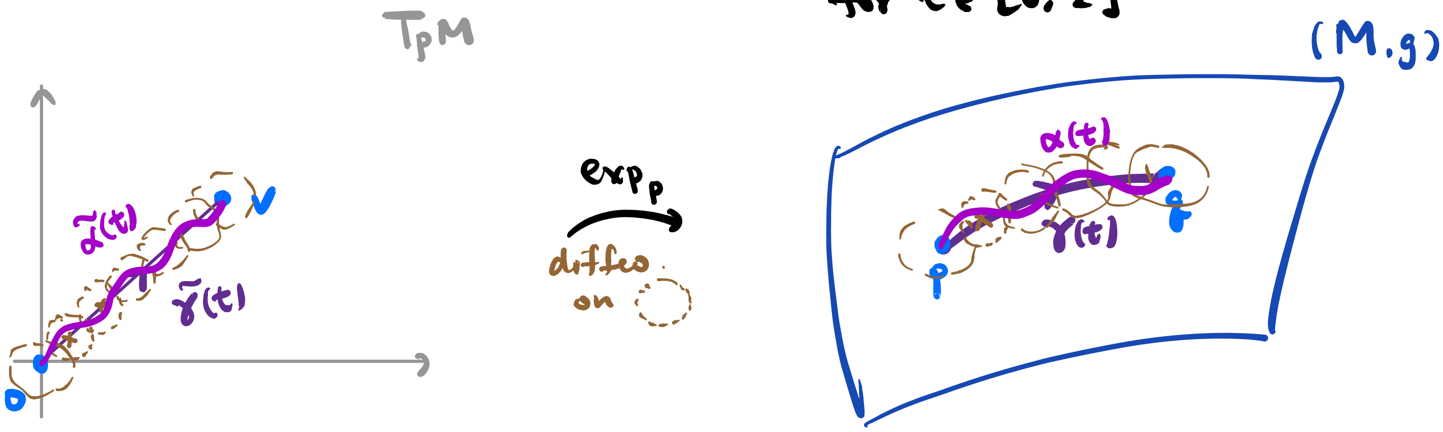
(ii) If $\exists\, t_0 \in (0,1)$ s.t. $\gamma(t_0)$ is conjugate to $p = \gamma(0)$,

then $\exists$ variation γ_s, fixing the end points p, q, st

$$L(\gamma_s) < L(\gamma) \quad \forall \text{ small } s \in (-\varepsilon, \varepsilon)$$

"**Sketch of Proof**": Let's start with (i).

By previous Prop. + hypothesis $\Rightarrow$ $\exp_p$ is a local diffeo. at each tv for $t \in [0,1]$



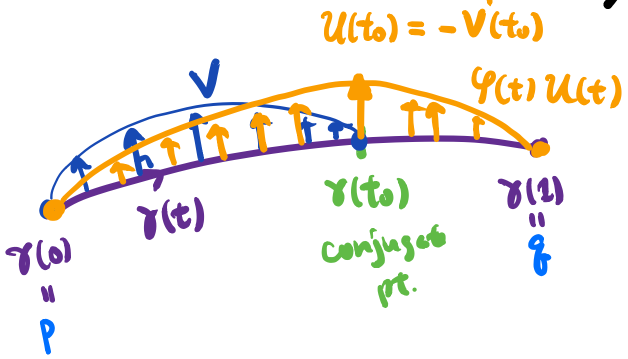
Gauss lemma $\Rightarrow$ $L(\alpha) \geq L(\gamma)$

Now, let's assume the hypothesis in (ii).

$\Rightarrow$ $\exists$ non-trivial Jacobi field $0 \neq V(t)$, $t \in [0, t_0]$

s.t. $V(0) = 0 = V(t_0)$

Note: $V'(0) \neq 0$



Claim: $\exists$ v.f. $W(t)$ along γ, $t \in [0,1]$, s.t. $I(W,W) < 0$

($\Rightarrow$ variation γ_s corresponding to W satisfy the conclusion)

Let $U(t)$ be a parallel v.f. along γ, $t \in [0,1]$, s.t.

$$U(t_0) = -V'(t_0)$$

Fix a smooth cutoff fcn $\varphi(t) : [0,1] \to \mathbb{R}$ s.t. $\begin{cases} \varphi(0) = \varphi(1) = 0 \\ \varphi(t_0) = 1 \end{cases}$

Define: For each $\alpha \in \mathbb{R}$, define (piecewise smooth) v.f. along γ

$$W = W_\alpha(t) := \begin{cases} V(t) + \alpha\, \varphi(t)\, U(t) & \text{if } t \in [0, t_0] \\ \alpha\, \varphi(t)\, U(t) & \text{if } t \in [t_0, 1] \end{cases}$$

$$I(W,W)$$

$$= \int_0^{t_0} \langle W', W' \rangle - \langle R(\gamma', W)\gamma', W \rangle \, dt + \int_{t_0}^{1} \langle W', W' \rangle - \langle R(\gamma', W)\gamma', W \rangle \, dt$$

$= 0$ $\because$ V is Jacobi field $V(0) = 0 = V(t_0)$

$$= \int_0^{t_0} \langle V', V' \rangle - \langle R(\gamma', V)\gamma', V \rangle \, dt$$

$$+ 2\alpha \int_0^{t_0} \langle V', \varphi' u \rangle - \langle R(\gamma', V)\gamma', \varphi u \rangle \, dt$$

$$+ \alpha^2 \int_0^{1} (\varphi')^2 \|u\|^2 - \langle R(\gamma', \varphi u)\gamma', \varphi u \rangle \, dt$$

$$= 2\alpha \underbrace{\langle V', \varphi u \rangle \Big|_{t=0}^{t=t_0}}_{= -\|V'(t_0)\|^2 < 0} + \alpha^2 I(\varphi u, \varphi u) < 0 \quad \text{for small } \alpha.$$

$\blacksquare$

Riemannian manifold as a metric space

(M^n, g) : Riemannian manifold (connected)

Define a distance on M as follow: for any $p, q \in M$.

$$d(p,q) := \inf \left\{ L(\gamma) : \begin{array}{l} \gamma : [0,1] \to M \text{ piecewise smooth} \\ \text{st } \gamma(0) = p, \ \gamma(1) = q \end{array} \right\}$$

FACT: (M, d) is a metric space.

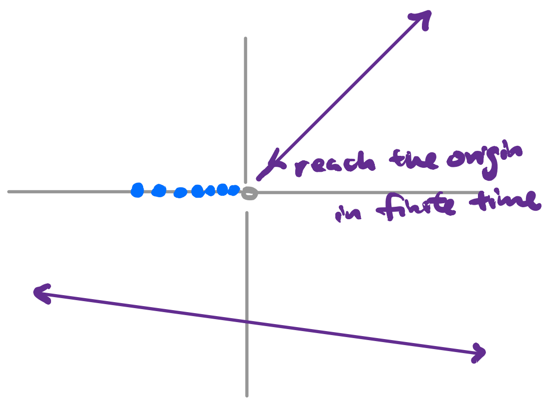
Q: When is it "complete" as metric space?

In fact, there is a more "differential-geometric" notion of "completeness".

<u>Def$^{\underline{n}}$</u>: A Riem. mfd (M^n, g) is **geodesically complete** if any geodesic on M can be infinitely extended on both sides (i.e. defined on all of $\mathbb{R}$).

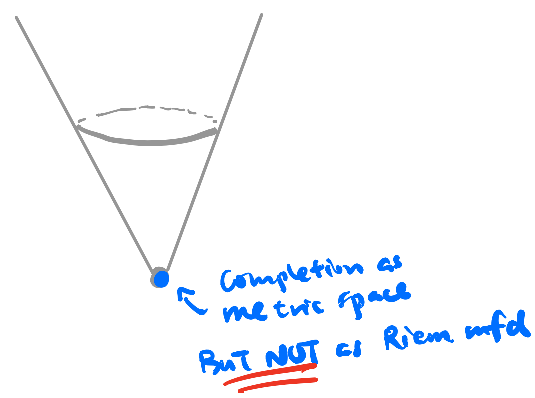
<u>Examples</u>:

(1) $\mathbb{R}^2 \setminus \{0\}$ w/. flat metric



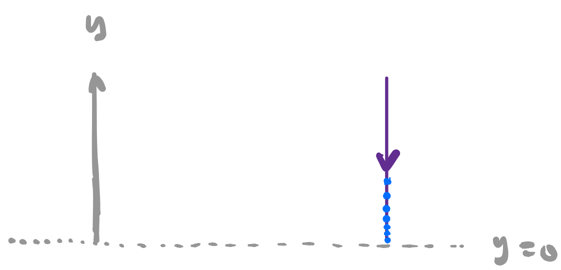
NOT complete as metric space
NOT geodesically complete

(2) $M^2 = \text{cone} \setminus \{\text{tip}\}. \subseteq \mathbb{R}^3$



NOT complete as metric space
NOT geodesically complete

(3). $\mathbb{R}^2_+ := \{ (x, y) \in \mathbb{R}^2 \mid y > 0 \}$.



NOT (geodesically) complete w.r.t. g_{flat}

<u>BUT</u> (geodesically) complete w.r.t

hyperbolic metric $g_{hyp} := \frac{1}{y^2}(dx^2 + dy^2)$

Hopf-Rinow Theorem: Let (M^n, g) be a smooth Riem mfd.

THEN, the following are equivalent:

(1) (M^n, g) is geodesically complete

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \parallel$

(2) (M, d) is complete as a metric space

(3) The exponential map at p, $\exp_p$, is well-defined on the whole T_pM, for **SOME** $p \in M$.

(4) The exponential map at p, $\exp_p$, is well-defined on the whole T_pM, for **ALL** $p \in M$.

If any of the above holds, then

$$(*) \left[\begin{array}{l} \forall\, p, q \in M,\ \exists \text{ minimizing geodesic } \gamma : [0,1] \to M \\ \quad \text{s.t. } \gamma(0) = p,\ \gamma(1) = q \quad \text{and} \quad L(\gamma) = d(p,q). \end{array} \right]$$

Proof: HW Exercise!